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Non-Abelian phase spaces

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Abstract. For a vector space V, its phase space T^*V is the vector space $V \oplus V^*$ together with the canonical symplectic form on it. Since the vector space is the same as an Abelian Lie algebra, the natural question is: given a Lie algebra $\mathcal{G}$, does there exist a phase space $T^*\mathcal{G}$? In general, the answer is negative. Below, for a large class of Lie algebras $\mathcal{G}$, the phase space $T^*\mathcal{G}$ is constructed. Three examples are treated in detail: $\mathcal{G} = gl(V)$; $\mathcal{G} = \mathcal{D}(\mathbf{R}^n)$, the Lie algebra of vector fields on $\mathbf{R}^n$; and current algebras.

1. Introduction

Let V be a vector space over a field k. The phase space of V, T^*V, is the vector space $V \oplus V^*$ endowed with the symplectic form

$$\omega(u \oplus u^*, v \oplus v^*) = \langle u^*, v\rangle - \langle v^*, u\rangle \qquad u, v \in V \quad u^*, v^* \in V^* . \tag{1}$$

If we consider V as an *Abelian* Lie algebra, the natural framework for the notion of the phase is the following: given a Lie algebra $\mathcal{G}$ (over k), make the vector space $\mathcal{G} + \mathcal{G}^*$ into a Lie algebra, $T^*\mathcal{G}$, in such a way that the symplectic form ω (equation (1)) is a 2-cocycle on $T^*\mathcal{G}$, i.e.

$$\omega([u_1 + u_1^*, u_2 + u_2^*], u_3 + u_3^*) + \text{CP} = 0 \qquad u_i \in \mathcal{G} \quad u_i^* \in \mathcal{G}^* \tag{2}$$

where 'CP' stands for 'cyclic permutation'. As posed for an arbitrary Lie algebra, this problem has, in general, no solution. Below, I will describe a large class of Lie algebras for which one can find a phase space. This class includes: $gl(V)$; $\mathcal{D}(\mathbf{R}^n)$, the Lie algebra of vector fields on $\mathbf{R}^n$; and current algebras. The basic idea is this: if the Lie algebra $\mathcal{G}$ comes out of an associative algebra A then everything is fine. But the associativity condition on A can be significantly weakened, producing the so-called quasi-associative algebras.

2. Quasi-associative algebras

Let A be an algebra over k, not necessarily associative, A is called *quasi-associative* [1] if

$$a(bc) - (ab)c = b(ac) - (ba)c \qquad \forall a, b, c \in A . \tag{3}$$

In particular, every associative algebra is quasi-associative.

Denote by Lie(A) the vector space of A with the new multiplication

$$[a, b] = ab - ba . \tag{4}$$

Proposition 1. If A is quasi-associative then Lie(A) is a Lie algebra.

Proof. We have to check the Jacobi identity

$$\big[[a,b],c\big] + \text{CP} = 0. \tag{5}$$

We have

$$\begin{aligned}\big[[a,b],c\big] + \text{CP} &= [ab - ba, c] + \text{CP}\\ &= \big[(ab-ba)c - c(ab-ba)\big] + \text{CP}\\ &= \big[(ab)c + \text{CP}\big] - \big[(ba)c + \text{CP}\big] - \big[c(ab) + \text{CP}\big] + \big[c(ba) + \text{CP}\big]\\ &= \big[(ab)c + \text{CP}\big] - \big[(ba)c + \text{CP}\big] - \big[a(bc) + \text{CP}\big] + \big[b(ac) + \text{CP}\big]\\ &= \Big(\big[(ab)c - a(bc)\big] - \big[(ba)c - b(ac)\big]\Big) + \text{CP} = 0 \qquad \text{by (3).}\end{aligned}$$

Remark 1. When A is associative rather than quasi-associative, the proposition 1 describes the standard fact and leads to the notion of the universal enveloping algebra of a Lie algebra. Formula (3) (or formula (3′) below) shows how to generalize this notion.

Remark 2. Proposition 1 remains true if one defines quasi-associativity using the opposite multiplication in A, so that the defining relation (3) becomes

$$(ab)c - a(bc) = (ac)b - a(cb) \qquad \forall a,b,c \in A. \tag{3′}$$

Remark 3. The defining relation (3) can be equivalently stated as

$$a(bc) - (ab)c \quad \text{is symmetric in } a, b \tag{6}$$

or as

$$a(bc) + (ba)c \quad \text{is symmetric in } a, b. \tag{7}$$

Example 1. Let $A = C^\infty(\mathbf{R}^1)$, the space of smooth functions on $\mathbf{R}^1$, with the multiplication

$$a \circ b = ab' \tag{8}$$

where $b' = \mathrm{d}b/\mathrm{d}x$. Then for the Lie algebra Lie(A) we have the commutator

$$[a,b] = a \circ b - b \circ a = ab' - a'b$$

so that Lie$(A) = \mathcal{D}(\mathbf{R}^1)$. On the other hand,

$$a \circ (b \circ c) - (a \circ b) \circ c = a(bc')' - (ab')c' = abc''$$

is symmetric in a, b, so that, by (6), A is quasi-associative.

Example 2. $A = K^n$, $K = C^\infty(\mathbf{R}^n)$, with the multiplication in A

$$(X \circ Y)^i = X^s Y^i_{,s} \qquad X = (X^i) \qquad Y = (Y^i) \in A \tag{9}$$

where $(\ )_{,s} = \partial(\cdot)/\partial x^s$, and $(x^1, \ldots, x^n)$ are the coordinates on $\mathbf{R}^n$; summation over repeated indices is in force. Since, for Lie(A), we have

$$[X,Y]^i = (X \circ Y)^i - (Y \circ X)^i = X^s Y^i_{,s} - Y^s X^i_{,s}$$

we see that Lie$(A) = \mathcal{D}(\mathbf{R}^n)$. To check the quasi-associativity of A, we have

$$\begin{aligned}\big[X \circ (Y \circ Z) - (X \circ Y) \circ Z\big]^i &= X^\alpha (Y \circ Z)^i_{,\alpha} - (X \circ Y)^s Z^i_{,s}\\ &= X^\alpha (Y^s Z^i{}_{,s})_{,\alpha} - X^\alpha Y^s_{,\alpha} Z^i_{,s} = X^\alpha Y^s Z^i_{,\alpha s}\end{aligned}$$

and this is symmetric in X, Y; by (6), A is quasi-associative.

3. T^*A, the phase space of A

Let $A^* = Hom(A, k)$ be the dual space to A. We shall make the vector space $T^*A = A + A^*$ into an algebra by extending the multiplication from A into T^*A via the rules

$$A^*A^* = \{0\} \qquad A^*A \subset A^* \qquad AA^* \subset A^* \tag{10a}$$

$$a^*b = 0 \tag{10b}$$

$$\langle ab^*, c\rangle = -\langle b^*, ac\rangle \qquad a^*, b^* \in A^* \qquad a, b, \in A\,. \tag{10c}$$

Proposition 2. If A is a quasi-associative algebra then the phase space of A, T^*A, is again a quasi-associative algebra.

Proof. Because of (10a), in checking the defining relation (3) for T^*A, we have to verify this relation only for the three cases when one of the arguments in this relation belongs to A^* and two others belong to A. We have,

(i) $a^*(bc) - (a^*b)c = b(a^*c) - (ba^*)c = 0$ by (10b);
(ii) similarly for the case when $b \in A^*$ in (3);
(iii) $\langle a(bc^*) - (ab)c^*, d\rangle$ (by (10c))

$$\begin{aligned} &= -\langle bc^*, ad\rangle + \langle c^*, (ab)d\rangle \quad \text{(by (10}c\text{))} \\ &= \langle c^*, b(ad)\rangle + \langle c^*, (ab)d\rangle = \langle c^*, b(ad) + (ab)d\rangle \end{aligned} \tag{11}$$

and, by (7), this is symmetric in a, b. Thus, by (6), T^*A is quasi-associative.

Remark 4. T^*A is not associative even if A is (otherwise the expression (11) would have been identically zero).

Remark 5. For the opposite multiplication (3′), formulae (10b) and (10c) take the form

$$ab^* = 0 \qquad \langle a^*b, c\rangle = -\langle a^*, cb\rangle\,. \tag{12}$$

For the commutator in the Lie algebra $T^*\mathrm{Lie}(A) := \mathrm{Lie}(T^*A)$ we have, by (10):

$$\left[\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2^* \end{pmatrix}\right] = \begin{pmatrix} a_1a_2 - a_2a_1 \\ a_1a_2^* - a_2a_1^* \end{pmatrix} \qquad a_1, a_2 \in A \qquad a_1^*, a_2^* \in A^*\,. \tag{13}$$

We see that $\mathrm{Lie}(T^*A) \approx \mathrm{Lie}(A) \ltimes A^*$, the semidirect sum, based on the representation $a : a^* \to a \circ a^*$, which is *not* a coadjoint representation of $\mathrm{Lie}(A)$ on $[\mathrm{Lie}(A)]^*$.

Proposition 3. The skewsymmetric symplectic form

$$\omega\left(\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2^* \end{pmatrix}\right) = \langle a_1^*, a_2\rangle - \langle a_2^*, a_1\rangle \tag{14}$$

is a 2-cocycle on the Lie algebra $\mathrm{Lie}(T^*A)$.

Proof. We have to check the cocycle condition (2). We have

$$\begin{aligned} &\omega\left(\left[\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2^* \end{pmatrix}\right], \begin{pmatrix} a_3 \\ a_3^* \end{pmatrix}\right) + \mathrm{CP} \quad \text{(by (13))} \\ &\quad = \omega\left(\begin{pmatrix} a_1a_2 - a_2a_1 \\ a_1a_2^* - a_2a_1^* \end{pmatrix}, \begin{pmatrix} a_3 \\ a_3^* \end{pmatrix}\right) + \mathrm{CP} \quad \text{(by (14))} \\ &\quad = (\langle a_1a_2^* - a_2a_1^*, a_3\rangle - \langle a_3^*, a_1a_2 - a_2a_1\rangle) + \mathrm{CP} \quad \text{(by (10}c\text{))} \\ &\quad = -(\langle a_2^*, a_1a_3\rangle + \mathrm{CP}) + (\langle a_1^*, a_2a_3\rangle + \mathrm{CP}) - (\langle a_3^*, a_1a_2 - a_2a_1\rangle + \mathrm{CP}) \\ &\quad = -(\langle a_3^*, a_2a_1\rangle + \mathrm{CP}) + (\langle a_3^*, a_1a_2\rangle + \mathrm{CP}) - (\langle a_3^*, a_1a_2 - a_2a_1\rangle) + \mathrm{CP} = 0\,. \end{aligned}$$

Remark 6. If $\mathcal{G}$ is a Lie algebra then, in general, the symplectic form on $\mathcal{G} \ltimes \mathcal{G}^*$ is *not* a 2-cocycle unless $\mathcal{G}$ is Abelian.

Proposition 4. The definition (10) of T^*A is natural.

Proof. Let $\tilde{A}$ be another quasi-associative algebra, and let $\varphi : A \to \tilde{A}$ be an isomorphism. Then the dual map $\varphi^* : \tilde{A}^* \to A^*$ is invertible. Denote by φ the map φ^{*-1}, so that

$$\langle \varphi(a^*), \varphi(b) \rangle = \langle a^*, b \rangle \qquad a^* \in A^*, b \in A. \tag{15}$$

We have to verify that the map φ preserves (10). This is obvious for (10a) and (10b). For the product AA^* we have

$$\begin{aligned} \langle \varphi(ab^*), \varphi(c) \rangle \quad (\text{by } (15)) \quad &= \langle ab^*, c \rangle \quad (\text{by } (10c)) \\ &= -\langle b^*, ac \rangle \quad (\text{by } (15)) \quad = -\langle \varphi(b^*), \varphi(ac) \rangle \quad (\text{since } \varphi \text{ is an isomorphism}) \\ &= -\langle \varphi(b^*), \varphi(a)\varphi(c) \rangle \quad (\text{by } (10c)) \quad = \langle \varphi(a)\varphi(b^*), \varphi(c) \rangle \end{aligned}$$

so that

$$\varphi(ab^*) = \varphi(a)\varphi(b^*).$$

Corollary 1. We have isomorphisms of Lie algebras

$$\text{Lie}(A) \approx \text{Lie}(\tilde{A}) \qquad \text{Lie}(T^*A) \approx \text{Lie}(T^*\tilde{A}).$$

Remark 7. Formula (15) shows that the symplectic 2-cocycles (14) map into each other under the isomorphism

$$\varphi : \text{Lie}(T^*A) \longrightarrow \text{Lie}(T^*\tilde{A}).$$

Remark 8. The construction of $T^*\mathcal{G}$ for $G = \text{Lie}(A)$ can be viewed from the following general perspective. Suppose P is a Poisson manifold. In general, only rarely can the phase space T^*P be made into a Poisson manifold in such a way that the Poisson bracket on T^*P extends that of P and is compatible with the canonical Poisson bracket on T^*P. When this is possible, P is called strongly Poisson [3]. The classical r-matrices and their nonlinear generalizations (the so-called Jacobi-ordered Poisson arrangements) can be interpreted in this language, but very little is known so far about which Poisson manifolds are strongly Poisson. Proposition 3 above, from this point of view, means that if A is quasi-associative then $P = [\text{Lie}(A)]^*$ is a strongly Poisson manifold.

Example 3. Let A be an associative algebra $End(V)$. Then $\text{Lie}(End(V)) = gl(V)$, with the commutator

$$[a_1, a_2] = a_1 a_2 - a_2 a_1.$$

Let us identify the dual space to $End(V)$, $[End(V)]^*$, with $End(V)$ itself by using the trace form

$$\langle a^*, a \rangle = \text{Tr}(a^* a).$$

By formula (10), the multiplication rules between A and A^* are

$$a^* \circ b = 0$$

and

$$\mathrm{Tr}\big((a \circ b^*)c\big) = \langle a \circ b^*, c\rangle = -\langle b^*, a \circ c\rangle = -\mathrm{Tr}(b^* ac)$$

so that

$$a \circ b^* = -b^* a \tag{16}$$

where on the right-hand side in (16) we have the usual matrix multiplication. Then (13) provides the following phase space, $T^* gl(V)$, to the Lie algebra $gl(V)$:

$$\left[\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2^* \end{pmatrix}\right] = \begin{pmatrix} a_1 a_2 - a_2 a_1 \\ -a_2^* a_1 + a_1^* a_2 \end{pmatrix}. \tag{17}$$

The symplectic 2-cocycle on this Lie algebra is

$$\omega\left(\begin{pmatrix} a_1 \\ a_1^* \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2^* \end{pmatrix}\right) = \mathrm{Tr}(a_1^* a_2 - a_2^* a_1). \tag{18}$$

Remark 9. All our constructions have finite-dimensional flavour. However, example 2 of $\mathcal{D}(\mathbf{R}^n)$ is an infinite-dimensional one and it can be properly described only in the language of variational calculus over differential algebras (see [2], part A). In this setting: $A = K^N$ where K is a commutative ring with n commuting derivations $\partial_1, \ldots, \partial_n : K \to K$; multiplication in A is given by differential operators; $A^* = \mathrm{Hom}(A, K)$ is identified with K^N; in formula (10c) the equality sign ' = ' is replaced by the equality modulo $\sum_s \mathrm{Im}\partial_s$ sign '$\sim$'; the same replacement takes place in (2), so that the symplectic form (14) becomes a generalized 2-cocycle on the Lie algebra Lie(T^*A); everything else remains unchanged. In the example 2, with $A = K^n$, $A^* = K^n = \{X^*\}$, formula (10c) yields

$$\begin{aligned}(X \circ X^*)_i Y^i &= \langle X \circ X^*, Y\rangle \sim -\langle X^*, X \circ Y\rangle \\ &= -X_i^* (X \circ Y)^i \quad \text{(by (9))} \quad = -X_i^* X^s Y^i_{,s} \sim (X_i^* X^s)_{,s} Y^i\end{aligned}$$

so that

$$(X \circ X^*)_i = (X^s X_i^*)_{,s} \tag{19}$$

which implies that

$$[\mathcal{D}(\mathbf{R}^n)]^* = \bigoplus_{i=1}^{n} V_1(i) \tag{20}$$

where $V_1(i)$ is the ith copy of the one-dimensional $\mathcal{D}(\mathbf{R}^n)$-module of volume forms

$$\mathcal{D}(\mathbf{R}^n) \ni X = X^s \partial_s \longmapsto \partial_s X^s = X + \mathrm{div}(X). \tag{21}$$

Formula (19) now becomes

$$\big((X^s\partial_s)\circ X^*\big)_i = (\partial_s X^s)(X^*_i) \tag{21'}$$

which justifies (20). From (13) we find the commutator in the Lie algebra $T^*\mathcal{D}(\mathbf{R}^n)$,

$$\left[\begin{pmatrix} X \\ X^* \end{pmatrix}, \begin{pmatrix} Y \\ Y^* \end{pmatrix}\right] = \begin{pmatrix} [X, Y] \\ X\circ Y^* - Y\circ X^* \end{pmatrix}. \tag{22}$$

Recall that if $\mathcal{G}$ is a finite-dimensional Lie algebra then the dual space $\mathcal{G}^*$ to $\mathcal{G}$ has the natural structure of a Poisson manifold:

$$\{f, g\}(q) = \langle q, [\mathrm{d}f|_q, \mathrm{d}g|_q]\rangle \qquad f, g \in C^\infty(\mathcal{G}^*) \qquad q \in \mathcal{G}^*.$$

In local coordinates, this formula has the standard form

$$\{f, g\} = \frac{\partial f}{\partial q_i} B_{ij} \frac{\partial g}{\partial q_j}$$

where $B = (B_{ij})$ is the (Hamiltonian) matrix

$$B_{ij} = c^k_{ij} q_k$$

and c^k_{ij} are the structure constants of $\mathcal{G}$ in a chosen basis. When $\mathcal{G}$ is an infinite-dimensional Lie algebra (over differential or differential-difference ring), the matrix B is read off the commutator in $\mathcal{G}$ via the following definition (details and proof may be found in [2], part A):

$$q_k[X, Y]_k \sim \mathbf{X}^t B(\mathbf{Y}) \qquad \forall X,\ Y \in \mathcal{G}$$

where $[X, Y]_k$ is the kth component of the commutator in $\mathcal{G}$. The matrix B is Hamiltonian since the associated Poisson bracket

$$\{f, g\} = \left(\frac{\delta f}{\delta q}\right)^t B \left(\frac{\delta g}{\delta q}\right)$$

satisfies the Jacobi identity (modulo divergencies).

Applying this construction to the Lie algebra $T^*\mathcal{D}(\mathbf{R}^n)$ (equation (22)) and denoting the dual coordinates on it by $(\mathbf{M}, \boldsymbol{p}) = (M_i, p^i)$, we have

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{X}^* \end{pmatrix}^t B \begin{pmatrix} \mathbf{Y} \\ \mathbf{Y} \end{pmatrix} \sim M_i[X, Y]^i + p^i(X\circ Y^* - Y\circ X^*)_i$$

where $\mathbf{X}, \mathbf{Y}$, etc are elements of K^n considered as column-vectors, whence

$$B = \begin{matrix} \\ M_i \\ p^i \end{matrix} \begin{matrix} \qquad M_j \qquad\qquad\qquad p^j \\ \begin{pmatrix} M_j\partial_i + \partial_j M_i & -p^j_{,i} \\ p^i_{,j} & 0 \end{pmatrix} \end{matrix}. \tag{23}$$

Proposition 3 now amounts to the property of the matrix $B + b$ being Hamiltonian, where b is the symplectic matrix

$$b = \begin{matrix} \\ M_i \\ p^i \end{matrix} \begin{matrix} \quad M_j \quad p^j \\ \begin{pmatrix} 0 & \delta^j_i \\ -\delta^i_j & 0 \end{pmatrix} \end{matrix}. \tag{24}$$

Remark 10. The pair of matrices (23), (24) has been used in [3] to construct phase-space analogues of the Korteweg–de Vries KdV equation (for $n = 1$) and local integrable systems in $n + 2 - d$ for any n. Also, in [3], a super KdV equation was lifted into the $\mathbf{Z}_2$-graded phase space. This suggests that all our previous constructions can be generalized into the $\mathbf{Z}_2$-graded (and even more generally graded) domain by inserting various signs into the formulae (2), (3) and (10). This is left to the reader as an exercise.

4. Current algebras

Let $K = C^\infty(\mathbf{R}^n)$. If $\mathcal{G}$ is a finite-dimensional Lie algebra then $Aff(\mathcal{G})_n$, the current (or affine) algebra of $\mathcal{G}$, is the following Lie algebra:

$$\left[\begin{pmatrix} X_1 \\ f_1 \otimes a_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ f_2 \otimes a_2 \end{pmatrix}\right] = \begin{pmatrix} [X_1, X_2] \\ X_1(f_2) \otimes a_2 - X_2(f_1) \otimes a_1 + f_1 f_2 \otimes [a_1, a_2] \end{pmatrix}$$

where $X_1, X_2 \in \mathcal{D}(\mathbf{R}^n)$, $f_1, f_2 \in K$, $a_1, a_2 \in \mathcal{G}$, and $X(f) = X^i f_{,i}$.

Suppose $\mathcal{G}$ has a phase space (e.g., if $\mathcal{G} = gl(V)$). This means that there exists a representation $\rho : \mathcal{G} \to End(\mathcal{G}^*)$ such that, on the semidirect sum Lie algebra $\mathcal{G} \ltimes_\rho \mathcal{G}^*$, the symplectic form is a 2-cocycle. This is equivalent to the identity

$$\rho^d(a)(b) - \rho^d(b)(a) = [a, b] \qquad \forall a, b \in \mathcal{G} \tag{25}$$

where $\rho^d : \mathcal{G} \to End(\mathcal{G})$ is the representation dual to ρ.

Proposition 5. If $\mathcal{G}$ has a phase space then so does the corresponding current algebra.

Proof. We define $T^*Aff(\mathcal{G})_n$ by the rule

$$\left[\begin{pmatrix} X_1 \\ f_1 \otimes a_1 \\ X_1^* \\ g_1 \otimes a_1^* \end{pmatrix}, \begin{pmatrix} X_2 \\ f_2 \otimes a_2 \\ X_2^* \\ g_2 \otimes a_2^* \end{pmatrix}\right]$$

$$= \begin{pmatrix} [X_1, X_2] \\ X_1(f_2) \otimes a_2 - X_2(f_1) \otimes a_1 + f_1 f_2 \otimes [a_1, a_2] \\ X_1 \circ X_2^* - X_2 \circ X_1^* \\ \widetilde{X}_1(g_2) \otimes a_2^* - \widetilde{X}_2(g_1) \otimes a_1^* + f_1 g_2 \rho(a_1)(a_2^*) - f_2 g_1 \rho(a_2)(a_1^*) \end{pmatrix} \tag{26}$$

where $g_1, g_2 \in K$, $a_1^*, a_2^* \in \mathcal{G}^*$, $\widetilde{X}(g) = (X^i g)_{,i}$, $X_1^*, X_2^* \in [\mathcal{D}(\mathbf{R}^n)]^*$, and $X \circ X^*$ is given by (19). By (6.137) in [2], formula (26) defines a Lie algebra. It remains to check that the symplectic form

$$\omega(1, 2) = \langle X_1^*, X_2 \rangle - \langle X_2^*, X_1 \rangle + g_1 f_2 \langle a_1^*, a_2 \rangle - g_2 f_1 \langle a_2^*, a_1 \rangle \tag{27}$$

defines a 2-cocycle on $T^*Aff(\mathcal{G})_n$, where $\omega(1, 2)$ is a short-hand notation for the value of ω on the arguments written in long-hand in (26). We have

$$\begin{aligned} &\omega([1, 2], 3) + \text{CP} \\ &\quad = (\langle X_1 \circ X_2^* - X_2 \circ X_1^*, X_3 \rangle - \langle X_3^*, [X_1, X_2] \rangle) + \text{CP} && (28a) \\ &\qquad + (f_1 g_2 \langle \rho(a_1)(a_2^*), a_3 \rangle f_3 - f_2 g_1 \langle \rho(a_2)(a_1^*), a_3 \rangle f_3 \\ &\qquad - g_3 \langle a_3^*, [a_1, a_2] \rangle f_1 f_2) + \text{CP} && (28b) \\ &\qquad + (\widetilde{X}_1(g_2) f_3 \langle a_2^*, a_3 \rangle - \widetilde{X}_2(g_1) f_3 \langle a_1^*, a_3 \rangle \\ &\qquad - g_3 X_1(f_2) \langle a_3^*, a_2 \rangle + g_3 X_2(f_1) \langle a_3^*, a_1 \rangle) + \text{CP}. && (28c) \end{aligned}$$

The expression $(28a)$ is ~ 0 by remark 9. The expression $(28b)$ vanishes by the assumption on $\mathcal{G}$ encoded in (25). The remaining expression $(28c)$ transforms into

$$\begin{aligned} &(\widetilde{X}_2(g_3) f_1 \langle a_3^*, a_1 \rangle + \text{CP}) - (\widetilde{X}_2(g_1) f_3 \langle a_1^*, a_3 \rangle + \text{CP}) \\ &\qquad - (g_1 X_2(f_3) \langle a_1^*, a_3 \rangle + \text{CP}) + (g_3 X_2(f_1) \langle a_3^*, a_1 \rangle + \text{CP}) \\ &\quad = [\widetilde{X}_2(g_3) f_1 + X_2(f_1) g_3] \langle a_3^*, a_1 \rangle + \text{CP} - [\widetilde{X}_2(g_1) f_3 + g_1 X_2(f_3)] \langle a_1^*, a_3 \rangle + \text{CP} \\ &\quad = \left[(X_2^i g_3 f_1)_{,i} \langle a_3^*, a_1 \rangle - (X_2^i g_1 f_3)_{,i} \langle a_1^*, a_3 \rangle\right] + \text{CP} \sim 0. \end{aligned}$$

Remark 11. Taking $\mathcal{G}$ to be one-dimensional Abelian, we obtain the phase space to the Lie algebra of linear differential operators of order $\leqslant 1$ on $\mathbf{R}^n$.

5. Vector fields

In classical mechanics, if M is a configuration manifold and $X \in \mathcal{D}(M)$ is a vector field on it, i.e. a derivation of the ring $C^\infty(M)$ of smooth functions on M, then X can be uniquely lifted from M into the phase space T^*M to a vector field $\widehat{X}$, say. This lift has the property

$$\widehat{[X, Y]} = [\widehat{X}, \widehat{Y}] \tag{29}$$

for any $X, Y \in \mathcal{D}(M)$. Let us see what is an analogy of this property in the framework of quasi-associative algebras.

Let $X, Y \in \mathrm{Der}(A)$ be two derivations of A, i.e.

$$X(ab) = X(a)b + aX(b) \qquad \forall a, b \in A \tag{30}$$

and similarly for Y. Note that $\mathrm{Der}(A)$ is a Lie algebra with respect to the commutator

$$[X, Y] = XY - YX \qquad X, Y \in \mathrm{Der}(A)$$

irrespective of whether A is associative or not. If we denote by I_A the ideal generated by (3) in a free k-algebra spanned by the elements of A, then it's easy to see that $\mathrm{Der}(A)(I_A) \subset I_A$. If A is quasi-associative, we extend $\mathrm{Der}(A)$ into $\mathrm{Der}(T^*A)$ by the rule

$$\widehat{X}(A^*) \subset A^* : \ \langle \widehat{X}(a^*), b\rangle = -\langle a^*, X(b)\rangle . \tag{31}$$

Proposition 6. $\widehat{X} \in \mathrm{Der}(T^*A)$.

Proof. We have to check the Leibnitz formula (30) in T^*A. In view of (10*a*) and (10*b*), we only need to verify the relation

$$\widehat{X}(ab^*) = X(a)b^* + a\widehat{X}(b^*) . \tag{32}$$

We have

$$\langle \widehat{X}(ab^*), c\rangle \quad (\text{by } (31)) \quad = -\langle ab^*, X(c)\rangle \quad (\text{by } (10c)) \quad = \langle b^*, aX(c)\rangle \tag{33a}$$

$$\begin{aligned} \langle X(a)b^* + a\widehat{X}(b^*), c\rangle &= -\langle b^*, X(a)c\rangle - \langle \widehat{X}(b^*), ac\rangle \\ &= -\langle b^*, X(a)c\rangle + \langle b^*, X(ac)\rangle \quad (\text{by } (30)) \quad = \langle b^*, aX(c)\rangle \end{aligned} \tag{33b}$$

which is the same as (33*a*)

Proposition 7. The map $\widehat{\ }: \mathrm{Der}(A) \to \mathrm{Der}(T^*A)$ is a homomorphism of Lie algebras.

Proof. We have to verify (29) for $X, Y \in \mathrm{Der}(A)$. This amounts to showing that

$$\widehat{[X, Y]}(a^*) = [\widehat{X}, \widehat{Y}](a^*)$$

which is equivalent to

$$\langle \widehat{[X, Y]}(a^*), b\rangle = \langle [\widehat{X}, \widehat{Y}](a^*), b\rangle$$

which can be seen as follows:

$$\begin{aligned} \langle \widehat{[X, Y]}(a^*), b\rangle \quad (\text{by } (31)) \quad &= -\langle a^*, [X, Y](b)\rangle \\ &= -\langle a^*, XY(b) - YX(b)\rangle = \langle \widehat{X}(a^*), Y(b)\rangle - \langle \widehat{Y}(a^*), X(b)\rangle \\ &= -\langle \widehat{Y}\widehat{X}(a^*), b\rangle + \langle \widehat{X}\widehat{Y}(a^*), b\rangle = \langle [\widehat{X}, \widehat{Y}](a^*), b\rangle . \end{aligned}$$

Proposition 8. If $X \in \mathrm{Der}(A)$ then $X \in \mathrm{Der}(\mathrm{Lie}(A))$.

Proof. We have to check that the equality

$$X([a,b]) = [X(a), b] + [a, X(b)]$$

follows from (30). We have

$$\begin{aligned} X([a,b]) &= X(ab - ba) = X(a)b + aX(b) - X(b)a - bX(a) \\ &= X(a)b - bX(a) + aX(b) - X(b)a = [X(a), b] + [a, X(b)]\,. \end{aligned}$$

Corollary 2. If $X \in \mathrm{Der}(A)$ then $\widehat{X} \in \mathrm{Der}(\mathrm{Lie}(T^*A))$.

Remark 12. For any $a \in \mathrm{Lie}(A)$, $ad(a) \in \mathrm{Der}(\mathrm{Lie}(A))$. The same map $ad(a)$ acts on A. It's easy to see that $ad(a) \in \mathrm{Der}(A)$ iff A is associative.

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